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Invisible permutations and rook placements on a Ferrers board

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Abstract

In this paper we introduce invisible permutations and rook length polynomials. We prove a relationship between rook length polynomials and Garsia–Remmel polynomials. We give explicit formulas for both of them.

1. Preliminaries and main results

Let λ be a partition of some integer. Write $\lambda=(\lambda_1,\ldots,\lambda_m)$, where $\lambda_1\geqslant\cdots\geqslant\lambda_m>0$. Sometimes, it is convenient to write $\lambda=(1^{v_1}2^{v_2}\cdots n^{v_n})$, where v_i is the number of λ_j's which are equal to i. We view a Ferrers board F_λ of shape λ as a subarray of an m by n matrix, where $n=\lambda_1$ and the kth row has length λ_k for $1\leqslant k\leqslant m$. For example, if $\lambda=(3,1)$, then

$$F_\lambda=\begin{pmatrix} * & * & * \\ & & * \end{pmatrix}.$$

Sometimes we write

$F_\lambda=$  .

We assume, contrary to the usual convention, that F_λ is right justified; the reason for this is explained below. Let $M(F_\lambda)=M_{m,n}(F_\lambda)$ be the set of all m by n matrices $(a_{i,j})$ with $a_{i,j}$ in some field K such that $a_{i,j}=0$ for $(i,j)\notin F_\lambda$. Thus, for $\lambda=(3,1)$,

$$M(F_\lambda)=\left\{\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & 0 & a_{2,3}\end{pmatrix}\right\}.$$

[1] This is part of the author's Ph.D. thesis at University of Wisconsin-Madison under the supervision of Prof. Louis Solomon.

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Say that λ is *parabolic* of type $(\mu_1, \mu_2, \ldots, \mu_k)$ if $m=n$ and there exist positive integers $\mu_1, \mu_2, \ldots, \mu_k$ such that

$$M(F_\lambda)=\left\{\begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,k} \\ 0 & A_{2,2} & \cdots & A_{2,k} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{k,k} \end{pmatrix}\right\}, \tag{1}$$

where $A_{i,i}$ is a μ_i by μ_i submatrix, for $1 \leqslant i \leqslant k$. Thus, $\lambda=(4,3,3,1)$ is parabolic with $(\mu_1, \mu_2, \mu_3)=(1,2,1)$. This explains our use of the word parabolic: if λ is parabolic then the invertible elements in $M(F_\lambda)$ are a parabolic subgroup of $GL_n(K)$. It also explains why our Ferrers boards are right justified. An element of $M(F_\lambda)$ is a rook placement of shape λ if it is a (0, 1) matrix with at most one 1 in each row and column. The 1's correspond to nonattacking rooks on the board F_λ. If there are r rooks, the matrix has rank r and we say that the rook placement has rank r. If there exists some rook placement σ of rank r on F_λ, then $\lambda_k \geqslant r-(k-1)$ for $1 \leqslant k \leqslant r$.

Historically, people considered rook placements as permutations with restricted positions. On the early studies of this topic, there is a survey in Riordan's book [16, Ch. 7]. Let C be a chess board with some forbidden positions and α_r be the number of ways to put r nonattacking rooks on this board. Then, the polynomial defined by

$$R(x, C)=\sum_{r \geqslant 0} \alpha_r x^r \tag{2}$$

is called the rook polynomial of the board C.

Definition 1.1. If $R(x, C)=R(x, C')$ for two given chess boards C and C', we say that these two boards are equivalent and write $C \sim C'$.

Riordan [16, p. 181] asked: when are two boards equivalent? The first major progress on this was made in the work of Foata and Schützenberger [3] on rook placements on a Ferrers board in 1970. A Ferrers board is said to be increasing if the heights of the columns from left to right increase strictly. In fact, Foata and Schützenberger proved the following theorem.

Theorem 1.2. *Every Ferrers board is equivalent to a unique increasing Ferrers board.*

From 1975 to 1978, Goldman, Joichi, Reiner and White published a series of papers on rook polynomials [6–10]. In particular, in [6], they found a very simple algorithm for deciding rook equivalence of Ferrers boards. This enabled them to count the number of boards in each of the equivalence classes. In 1986, Garsia and Remmel considered a q-analogue of the rook numbers α_r which we call the Garsia–Remmel rook polynomial. We define this polynomial in Definition 1.7.

Definition 1.3. Let r, m, n be nonnegative integers such that $r \leqslant \min\{m, n\}$. Let $\Omega^r_{m,n}$ be the set of all integer sequences

$$(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = (u_1, u_2, \ldots, u_{n-r}, v_1, v_2, \ldots, v_r, w_1, w_2, \ldots, w_{m-r})$$

satisfying

$$\begin{aligned} &0 \leqslant u_1 \leqslant u_2 \leqslant \cdots \leqslant u_{n-r} \leqslant r, \\ &0 \leqslant w_{m-r} \leqslant w_{m-r-1} \leqslant \cdots \leqslant w_1 \leqslant r, \\ &0 \leqslant v_i \leqslant i-1, \ 1 \leqslant i \leqslant r. \end{aligned} \tag{3}$$

We call the sequences satisfying these inequalities *inversion sequences*. When $m = n = r$, $\boldsymbol{u}$ and $\boldsymbol{w}$ do not exist and the sequence $\boldsymbol{v}$ defined above is the inversion sequence of an ordinary permutation; see [11]. We write $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$ as $(\boldsymbol{u}|\boldsymbol{v}|\boldsymbol{w})$ to mark the position between two consecutive components.

Let R^r_λ be the set of all the rook placements of rank r on a Ferrers board F_λ. For a rectangular board, we sometimes write $R^r_{n^m}$ as $R^r_{m,n}$ to match the notation $\Omega^r_{m,n}$. Note that if there exists some rook placement σ of rank r on F_λ, then

$$\lambda_k \geqslant r-(k-1) \quad \text{for } 1 \leqslant k \leqslant r. \tag{4}$$

If $r \geqslant 1$ let

$$v_r = \sum_{i=1}^{r} E_{i,n-r+i}, \tag{5}$$

where $E_{i,j}$ is the matrix with 1 at (i,j) and 0's elsewhere. Clearly, λ satisfies (4) if and only if $v_r \in F_\lambda$. Let W_n be the symmetric group on $[n] = \{1, \ldots, n\}$. If $w \in W_n$ we identify w with the permutation matrix

$$w = \sum_{i=1}^{n} E_{i,w(i)}. \tag{6}$$

An inversion of w is a pair $((w(i), w(j)))$, where $i < j$ and $w(i) > w(j)$. Let

$$S(n) = \{(12), (23), \ldots, (n-1, n)\}$$

be the set of distinguished generators of W_n. Now, we introduce a length function l_λ on R^r_λ.

Definition 1.4. Let λ be a partition such that $v_r \in F_\lambda$. When $r = 0$, let $v_0 = 0$. For $\sigma \in R^r_\lambda$, the length function $l_\lambda(\sigma)$ is defined by

$$l_\lambda(\sigma) = \min\{k + h \mid \sigma = s_k \cdots s_1 v_r s'_1 \cdots s'_h\},$$

where $s_i \in S(m)$ and $s'_j \in S(n)$ and

$$s_p \cdots s_1 v_r s'_1 \cdots s'_q \in R^r_\lambda$$

for each $1 \leqslant p \leqslant k$ and $1 \leqslant p \leqslant h$.

Thus $l_\lambda(\sigma)$ is the minimum number of adjacent row and/or column transpositions required to get σ from $v_r = \sum_{i=1}^r E_{i,n-r+i}$, such that all the intermediate rook placements are in the board F_λ. This length function was first used as the length of rook matrices by Solomon in his work on the Iwahori ring of $M_n(\boldsymbol{F}_q)$ [18]. If $\sigma \in R^r_\lambda$ and $\sigma \in R^r_\mu$ for two different partitions λ and μ, it is not clear without proof that $l_\lambda(\sigma) = l_\mu(\sigma)$. We will prove this in Lemma 5.33. Until then we will assume that $l_\lambda(\sigma)$ is defined with respect to a rectangular board $\lambda = (n^m)$ and write $l(\sigma)$ for this length function.

For example, if $\lambda = (3,3,1)$ and $r = 2$, then

$$v_r = \begin{array}{|c|c|c|} \hline & 1 & \\ \hline & & 1 \\ \hline \multicolumn{2}{c|}{} & \\ \cline{3-3} \end{array}\,.$$

Take

$$\sigma = \begin{array}{|c|c|c|} \hline & & \\ \hline 1 & & \\ \hline \multicolumn{2}{c|}{} & 1 \\ \cline{3-3} \end{array}\,.$$

Then, it is easy to check that the minimum number of adjacent row and/or column transpositions required to get σ from v_r is 3. For example we can first interchange the first row and the second row. Then interchange the first column and the second column. At the end interchange the second row and the third row to get v_r. Here we use the assumption in the definition of Ferrers boards that a Ferrers board is a partial array of a matrix in which all the entries outside of the board are zeros.

If $\lambda = (n^m)$ and $r = n = m$, then $R^r_\lambda = W_n$ the symmetric group on n letters. In this case the function l agrees with the usual length function on the symmetric group W_n in terms of the generators $S(n)$. It is equal to the number of inversions in a permutation. Garsia and Remmel [4] considered another numerical function on R^r_λ.

Definition 1.5. For each $\sigma \in R^r_\lambda$, place a dot in every position that is above a rook or to the right of a rook and a circle in each of the remaining positions of F_λ. Let $\mathrm{GR}(\sigma)$ denote the number of circles.

Example 1.6. Suppose $\lambda = (4,3,2)$ and $\sigma = E_{12} + E_{33}$. Then we get the configuration

$$\begin{array}{cccc} \circ & 1 & \bullet & \bullet \\ & \circ & \bullet & \circ \\ & & 1 & \bullet \end{array}$$

Thus, $GR(\sigma) = 3$.

Definition 1.7. Let

$$R_r(\lambda, q)=\sum_{\sigma\in R_\lambda^r} q^{GR(\sigma)},$$

and

$$RL_r(\lambda, q)=\sum_{\sigma\in R_\lambda^r} q^{l(\sigma)}.$$

Call them Garsia–Remmel rook polynomial and rook length polynomial, respectively.

Clearly, when $q=1$, we have $R_r(\lambda,1)=RL_r(\lambda,1)=\alpha_r$, where α_r is as in (2). So both of the polynomials defined above are q-analogues of the rook number α_r of the board $C=F_\lambda$.

Definition 1.8. Let $(k)_q=1+q+\cdots+q^{k-1}$. Define $[k]!_q=(1)_q(2)_q\cdots(k)_q$. The Gaussian binomial coefficient is defined by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q=\frac{[n]!_q}{[m]!_q[n-m]!_q}. \tag{7}$$

In [4, Theorem 1.1] there is the following very useful recurrence formula.

Theorem 1.9. *Let $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_m)$ and suppose $m'\geqslant m$. Let*

$$\lambda+1^{m'}:=(\lambda_1+1,\ldots,\lambda_m+1,\underbrace{1,\ldots,1}_{m'-m}),$$

i.e., $F_{\lambda+1^{m'}}$ is the Ferrers board obtained by adjoining a column of length m' to the board F_λ. Then

$$R_r(\lambda+1^{m'},q)=q^{m'-r}R_r(\lambda,q)+(m'-r+1)_qR_{r-1}(\lambda,q).$$

Definition 1.10. A sequence $(a_i)_{1\leqslant i\leqslant m}$ of real numbers is unimodal if there is some $k\in[m]$ such that $a_1\leqslant\cdots\leqslant a_k\geqslant\cdots\geqslant a_m$.

By means of the formula in Theorem 1.9, Garsia and Remmel computed a number of interesting examples which led to the following conjecture.

Conjecture 1.11 (Garsia–Remmel). The coefficient sequence of a Garsia–Remmel polynomial is unimodal.

The main results of this section are the following three theorems.

Theorem 1.12. *There is a bijection* $\Phi: R^r_{m,n} \to \Omega^r_{m,n}$ *such that if* $\Phi(\sigma)=(\boldsymbol{u}|\boldsymbol{v}|\boldsymbol{w})$ *then*

$$l(\sigma)=\sum_{i=1}^{n-r} u_i+\sum_{j=1}^{r} v_j+\sum_{k=1}^{m-r} w_k. \tag{8}$$

For given σ the corresponding triple $(\boldsymbol{u}|\boldsymbol{v}|\boldsymbol{w})=(\boldsymbol{u}(\sigma)|(\boldsymbol{v}(\sigma)|\boldsymbol{w}(\sigma))$ may be described as follows: Suppose $\sigma=E_{c_1,b_1}+\cdots+E_{c_r,b_r}$, where $c_1<c_2<\cdots<c_r$. Call the rook at (c_i,b_i) the ith rook. Then,

- u_i is the number of rooks to the left of the ith zero column in σ,
- v_i is the number of rooks above and to the right (to the 'northeast') of the ith rook in σ,
- w_i is the number of rooks below the ith zero row in σ.

If λ is a partition let

$$C_{\lambda,r}=\sum_{i=1}^{m}\lambda_i-\frac{r(r+1)}{2}. \tag{9}$$

Theorem 1.13. *If* $\sigma\in R^r_\lambda$, *then*

$$GR(\sigma)+l(\sigma)=C_{\lambda,r} \tag{10}$$

and thus

$$R_r(\lambda,q)=q^{C_{\lambda,r}}RL_r(\lambda,q^{-1}). \tag{11}$$

Theorem 1.14 (Formula of Rook Length Polynomials). *If* λ *is a partition with* m *parts then*

$$RL_r(\lambda,q)=\sum_{1\leqslant i_1<\cdots<i_r\leqslant m} q^{\sum_{j=1}^{r}(i_j-j)}\prod_{j=1}^{r}(\lambda_{i_j}-r+j)_q, \tag{12}$$

where $(k)_q=1+q+q^2+\cdots+q^{k-1}$.

These two theorems give the following explicit formula for the Garsia–Remmel polynomial.

Corollary 1.15.

$$R_r(\lambda,q)=q^{C_{\lambda,r}}\sum_{1\leqslant i_1<\cdots<i_r\leqslant m} q^{-\sum_{j=1}^{r}(i_j-j)}\prod_{j=1}^{r}(\lambda_{i_j}-r+j)_{q^{-1}}. \tag{13}$$

Corollary 1.16. *For an* m *by* n *rectangular board, we have*

$$RL_r(n^m,q)=\begin{bmatrix} m \\ r \end{bmatrix}\begin{bmatrix} n \\ r \end{bmatrix}[r]! \tag{14}$$

If $m=n$, this is proved in [18] by means of the root system of type A_{n-1}.

Corollary 1.17. *If $r=m\leqslant n$, then*

$$RL_m(\lambda,q)=(\lambda_1-m+1)_q(\lambda_2-m+2)_q\cdots(\lambda_m)_q. \tag{15}$$

Definition 1.18. A sequence $\{a_i\}_{1\leqslant i\leqslant m}$ of numbers is symmetric if $a_i=a_{m-i+1}$ for $1\leqslant i\leqslant m$.

Remark. It follows from Corollary 1.16 that the sequence of coefficients of the rook length polynomial $RL_m(\lambda,q)$ is symmetric if $\lambda=(n^m)$. Also it follows from Corollary 1.17 that the sequence of coefficients of $RL_m(\lambda,q)$ is symmetric for any λ if $r=m$. The geometric significance of symmetry is, in view of the results of Section 3, that the cellular decomposition of the variety $B\backslash M_\lambda$ has a unique dense cell.

Corollary 1.19. *Suppose $r=m=n$ and λ is parabolic of type $(\mu_1,\ldots,\mu_k)$. Then*

$$RL_m(\lambda,q)=\prod_{i=1}^{k}[\mu_i]!. \tag{16}$$

In particular, if λ and λ' are parabolic of types $(\mu_1,\ldots,\mu_k)$ and $(\mu_1',\ldots,\mu_k')$, respectively, where $(\mu_1',\ldots,\mu_k')$ is a permutation of $(\mu_1,\ldots,\mu_k)$, then

$$RL_r(\lambda,q)=RL_r(\lambda',q).$$

2. A bijection: maximum rank case

The main idea in this section and the next is to extend a placement σ of r rooks on an m by n board to a placement $P(\sigma)$ of $m+n-r$ rooks on an $m+n-r$ by $m+n-r$ board. We identify $P(\sigma)$ with the corresponding permutation of $[m+n-r]$ and call $P(\sigma)$ the *invisible permutation* corresponding to σ. There are various ways to extend a rook placement σ to a permutation of $[m+n-r]$. The choice here is made so that $l(\sigma)$ is equal to the number of inversions in the permutation $P(\sigma)$.

In order to help the reader understand the definition of $P(\sigma)$ we give an example.

Example 2.20. Let $r=2$, $m=n=4$. Let

$$\sigma=E_{14}+E_{32}=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix}.$$

Thus the permutation matrix corresponding to σ is

$$P(\sigma)=(E_{11}+E_{23})+(E_{34}+E_{52})+(E_{45}+E_{66})$$

$$=\left(\begin{array}{cccc|cc} 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ \hline \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&0&0\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&1&0\\ \mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}&0&0\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&0&1 \end{array}\right).$$

The 4×4 submatrix of $P(\sigma)$ at the southwest corner is exactly the rook matrix σ.

The precise definition of the permutation $P(\sigma)$ is given in (2.21). Since it is hard to see through the notation, we give an informal definition of the corresponding permutation matrix. The matrix $P(\sigma)$ is $m+n-r$ by $m+n-r$ and has σ in its southwest corner. We call this southwest corner of $P(\sigma)$ the *real part* of $P(\sigma)$. We call the columns $n+1,\ldots,n+m-r$ and the rows $1,\ldots,m-r$ the *imaginary part* of $P(\sigma)$. In the example these are the columns to the right of the vertical line and the rows above the horizontal line. The 1's are placed in the imaginary part of $P(\sigma)$ so that the column indices increase as the row indices increase. This describes their positions uniquely. From the method of placement of the 1's in the imaginary part we see that $l(\sigma)$ is equal to the number of inversions in the permutation $P(\sigma)$ which is the usual length of the permutation. So we have

$$l(\sigma)=l(P(\sigma)). \tag{18}$$

This mapping $\sigma\to P(\sigma)$ allows us to translate problems about rook placements into problems about inversions of permutations. This is one of the main ideas in this section. It would also be possible to put the real part σ of $P(\sigma)$ in the northeast corner. But, if σ is placed in either of the other two corners, the combinatorics becomes messy. In the following we think of $P(\sigma)$ as a permutation or a permutation matrix whichever is convenient. The formal definition of the permutation $P(\sigma)$ is as follows.

Definition 2.21. Let $\sigma\in R^r_{n^m}$. Let $\sigma=\sum_{i=1}^r E_{c_i,b_i}$ with $c_1<c_2<\cdots<c_r$. Define a permutation $P(\sigma)\in W_{m+n-r}$ by

$$P(\sigma)=\left(\begin{array}{c|ccc|c} 1\cdots n-r & n-r+c_1 & \cdots & n-r+c_r & d_1\ \cdots\ d_{m-r}\\ a_1\cdots a_{n-r} & b_1 & \cdots & b_r & n+1\cdots n+m-r \end{array}\right),$$

where $\{a_1,a_2,\ldots,a_{n-r}\}$ is the complement of $\{b_1,b_2,\ldots,b_r\}$ in $[n]$ with $a_1<a_2<\cdots<a_{n-r}$, and $\{d_1,d_2,\ldots,d_{m-r}\}$ is the complement of $\{n-r+c_1,\ldots,n-r+c_r\}$ in $\{n-r+1,\ldots,n-r+m\}$ with $d_1<d_2<\cdots<d_{m-r}$.

Remark. From this definition, we see that the a_i's index the zero columns of σ and the d_i's index the zero rows of σ.

In Example 2.20, since $\sigma = E_{14} + E_{32}$, we have $c_1 = 1$, $c_2 = 3$, $b_1 = 4$ and $b_2 = 2$. Thus $\{a_1, a_2\} = [n] - \{b_1, b_2\} = [4] - \{4, 2\} = \{1, 3\}$. So $a_1 = 1$ and $a_2 = 3$. Similarly,

$$\begin{aligned}\{d_1, d_2\} &= [n+m-r] - [n-r] - \{n-r+c_1, n-r+c_2\}\\ &= \{3,4,5,6\} - \{2+c_1, 2+c_2\} = \{4,6\}.\end{aligned}$$

So $d_1 = 4$ and $d_2 = 6$. Thus

$$P(\sigma) = \left(\begin{array}{cc|cc|cc} 1 & 2 & 3 & 5 & 4 & 6 \\ 1 & 3 & 4 & 2 & 5 & 6 \end{array}\right).$$

Then the permutation matrix $P(\sigma)$ is

$$P(\sigma) = (E_{11} + E_{23}) + (E_{34} + E_{52}) + (E_{45} + E_{66}),$$

which is exactly the same as the one given in Example 2.20.

If we write $P(\sigma)$ as a two row array as in Definition 2.21, the columns to the left of the first vertical line are called the *column imaginary part* of $P(\sigma)$. The columns to the right of the second vertical line are called the *row imaginary part* of $P(\sigma)$. The columns between the two vertical lines are called the *real part* of $P(\sigma)$. Note in Example 2.20 that the 4×4 submatrix of $P(\sigma)$ at the southwest corner, the real part of $P(\sigma)$, is $E_{34} + E_{52}$ which corresponds to $\sigma = E_{14} + E_{32}$. There will always be this shift of row indices. If $\sigma = \sum_{i=1}^{r} E_{c_i, b_i}$ then the corresponding block in the southwest corner of $P(\sigma)$ is $\sigma = \sum_{i=1}^{r} E_{n-r+c_i, b_i}$. In this section we consider the special case $r = m \leqslant n$. The general case will be done in the next section. We call the elements of $R^m_{m,n}$ maximum rank rook matrices. If $r = m$, then $c_i = i$ for $i = 1, \ldots, m$ in Definition 2.21. Thus,

$$P(\sigma) = \left(\begin{array}{ccc|ccc} 1 & \cdots & n-m & n-m+1 & \cdots & n \\ a_1 & \cdots & a_{n-m} & b_1 & \cdots & b_m \end{array}\right). \tag{19}$$

The following theorem is the main result of this section. It is the special case $r = m$ of Theorem 1.12. If $r = m = n$, it is Hall's classical theorem [11] and the map in our theorem is Hall's bijection.

Theorem 2.22. *There exists a bijection $\boldsymbol{\Phi} : R^m_{m,n} \to \Omega^m_{m,n}$ such that if $\sigma \in R^m_{m,n}$ and $(\boldsymbol{u} \mid \boldsymbol{v}) = \boldsymbol{\Phi}(\sigma)$, then*

$$l(\sigma) = \sum_{i=1}^{n-m} u_i + \sum_{j=1}^{m} v_j. \tag{20}$$

Before we prove this theorem, we need a lemma which we state for arbitrary $r \leqslant m$ since we will use it again in the next section.

Lemma 2.23. *The cardinalities of $\Omega^r_{m,n}$ and $R^r_{m,n}$ are both equal to $\binom{m}{r}\binom{n}{r} r!$*

Proof. The fact that there are $r!$ choices for the sequence $\boldsymbol{v}$ is clear from (1.3). As for the sequence $\boldsymbol{u}$, consider the sequence $(u_1+1, u_2+2, \ldots, u_{n-r}+n-r)$. It satisfies the inequalities,

$$1 \leqslant u_1+1 < u_2+2 < \cdots < u_{n-r}+n-r \leqslant n.$$

So there are $\binom{n}{r}$ choices for this sequence. Since the sequences $(u_1+1, u_2+2, \ldots, u_{n-r}+n-r)$ are in one-to-one correspondence with the sequence $\boldsymbol{u}$, there are $\binom{n}{r}$ choices for the sequences $\boldsymbol{u}$. Similarly, there are $\binom{m}{r}$ choices for the sequence $\boldsymbol{w}$. Hence the cardinality of $\Omega^r_{m,n}$ is $\binom{m}{r}\binom{n}{r}r!$

On the other hand, an arbitrary rook placement from $R^r_{m,n}$ can be obtained as follows: choose r rows from m rows, choose r columns from n columns and then put r nonattacking rooks on an r by r board. So $|R^r_{m,n}| = \binom{m}{r}\binom{n}{r}r!$ $\square$

Proof of Theorem 2.22. Let $\sigma = \sum_{i=1}^m E_{i,b_i} \in R^m_{m,n}$. Write $P(\sigma) = (\boldsymbol{a}|\boldsymbol{b})$, where $\boldsymbol{a} = (a_1, \ldots, a_{n-r})$ and $\boldsymbol{b} = (b_1, \ldots, b_r)$ as in (19). First, note that the imaginary part $\boldsymbol{a}$ is increasing. Thus any inversion of $P(\sigma)$ which involves some a_i must be of the form (a_i, b_j). Let

$$u_i = |\{b_j \mid b_j < a_i\}| \quad \text{for } 1 \leqslant i \leqslant n-m, \tag{21}$$

$$v_i = |\{b_j \mid b_j < b_i \text{ and } j < i\}| \quad \text{for } 1 \leqslant i \leqslant m. \tag{22}$$

Thus we get an integer sequence

$$(\boldsymbol{u}|\boldsymbol{v}) = (u_1, u_2, \ldots, u_{n-m} \mid v_1, v_2, \ldots, v_m).$$

Define $\boldsymbol{\Phi}(\sigma) := (\boldsymbol{u}|\boldsymbol{v})$. Since the imaginary part of $P(\sigma)$ is increasing,

$$0 \leqslant u_1 \leqslant u_2 \leqslant \cdots \leqslant u_{n-m} \leqslant m. \tag{23}$$

Notice that for each i, the number of inversions of the form (b_j, b_i) is determined solely by the sign of each difference $b_j - b_i$ rather than the actual values of b_j and b_i. Thus the v_i's satisfy the inequalities

$$0 \leqslant v_k \leqslant k-1, \quad 1 \leqslant k \leqslant m. \tag{24}$$

(Note in case $m=n$ that these are exactly the restrictions on the inversion sequences of elements of W_m; see [11], or [2, p. 90]). Hence, the sequence $\boldsymbol{\Phi}(\sigma) = (\boldsymbol{u}|\boldsymbol{v})$ satisfies the inequalities in Definition 1.3. Thus, $\boldsymbol{\Phi}(\sigma) \in \Omega^m_{m,n}$.

Next, we need to verify that $\boldsymbol{\Phi}(\sigma)$ satisfies (20). Since $r=m$, σ has $n-m$ zero columns. By the definition of l, we can first move all the zero columns of σ to the left of all the nonzero columns, keeping the relative positions of the nonzero columns unchanged. This takes exactly u_i adjacent columns swaps for the ith zero column, where $i = 1, \ldots, n-m$. Then, the last m columns form an m by m permutation matrix, which has $\sum_{i=1}^m v_i$ inversions. Hence (20) is true. Thus $\boldsymbol{\Phi}$ is a mapping from $R^m_{m,n}$ to $\Omega^m_{m,n}$ which satisfies (20).

By Lemma 2.23, $|\Omega^m_{m,n}|=|R^m_{m,n}|$. Therefore, in order to show that $\boldsymbol{\Phi}$ is a bijection, we need only show that $\boldsymbol{\Phi}$ is surjective. Given $(\boldsymbol{u}|\boldsymbol{v})\in\Omega^m_{m,n}$ we will construct $\sigma\in R^m_{m,n}$ such that $\boldsymbol{\Phi}(\sigma)=(\boldsymbol{u}|\boldsymbol{v})$. To construct σ we will use the following algorithm which will also be used in the next section. For this reason, we use the parameter r instead of m though $r=m$ in this section.

Algorithm 2.24. Input $(\boldsymbol{u}|\boldsymbol{v})\in\Omega^r_{m,n}$.

(1) For $1\leqslant i\leqslant n-r$, let $a_i=u_i+i$,

(2) Define a sequence of integers $b_r, b_{r-1},\ldots,b_1$ and a sequence of subsets $B_r, B_{r-1},\ldots,B_1$ of $[n]$ recursively as follows: Let $B_r=[n]-\{a_1,\ldots,a_{n-r}\}$. Let b_r be the (v_r+1)th largest element in B_r. For $i\geqslant 1$, let $B_{r-i}=B_{r-i+1}-\{b_{r-i+1}\}$, let b_{r-i} be the $(v_{r-i}+1)$th largest element in B_{r-i} (here the 1st largest is the largest).

Output $\boldsymbol{a}=(a_1,a_2,\ldots,a_{n-r})$ and $\boldsymbol{b}=(b_1,b_2,\ldots,b_r)$.

Step (2) is possible since $v_i\leqslant i-1$, by Definition 1.3. By construction, for the given $(\boldsymbol{u}|\boldsymbol{v})$, $(\boldsymbol{a}|\boldsymbol{b})$ satisfies the conditions in (21) and (22). Clearly $\boldsymbol{a}$ is a strictly increasing sequence and $(\boldsymbol{a}|\boldsymbol{b})$ is a permutation of $[n]$. Define $\sigma=\sum_{i=1}^r E_{i,b_i}$. Then $P(\sigma)=(\boldsymbol{a}|\boldsymbol{b})$ by (23). Thus $\boldsymbol{\Phi}(\sigma)=(\boldsymbol{u}|\boldsymbol{v})$. $\square$

Note that the construction of Algorithm 2.24 gives the inverse of the mapping $\boldsymbol{\Phi}$. In order to illustrate the bijection given above, we consider the following example.

Example 2.25. Let

$$\sigma=\begin{pmatrix}0&0&0&1\\0&1&0&0\end{pmatrix}.$$

Then

$$P(\sigma)=\left(\begin{array}{cc|cc}1&2&3&4\\1&3&4&2\end{array}\right).$$

Thus $(\boldsymbol{a}|\boldsymbol{b})=(a_1,a_2\,|\,b_1,b_2)=(1,3\,|\,4,2)$. Consider first the inversions of the form (a_i,b_j). Since there is no b_i which is less than $a_1=1$ we have $u_1=0$. Since there is only one term $b_2=2$ in the real part of $P(\sigma)$ which is less than $a_2=3$, we have $u_2=1$. Next, look at the real part (b_1,b_2). There is no term b_j in the real part of $P(\sigma)$ which is to the right of $b_1=4$ and bigger than 4, so $v_1=0$. On the other hand, there is exactly one term $b_1=4$ in the real part of $P(\sigma)$ which is to the right of $b_2=2$ and bigger than 2, so $v_2=1$. Hence we have $\Phi(\sigma)=(u_1,u_2\,|\,v_1,v_2)=(0,1\,|\,0,1)$.

In the opposite direction, let $(u_1,u_2\,|\,v_1,v_2)=(0,1\,|\,0,1)$. Then $(a_1,a_2)=(u_1+1,u_2+2)=(1,3)$. Thus $B_2=\{1,2,3,4\}-\{a_1,a_2\}=\{2,4\}$. As $v_2=1$, b_2 is the $(1+1)$st largest element in B_2, so $b_2=2$. Thus $B_1=B_2-\{2\}=\{4\}$. Similarly, $v_1=0$ implies that b_1 is the $(0+1)$st largest element in B_1 so $b_1=4$. To recover σ from $P(\sigma)$, we chop off the first two rows of the permutation matrix $P(\sigma)$.

3. The bijection in general form

In this section, we remove the restriction $r=m$ and prove Theorem 1.12 in general. The idea here is similar to that used in the previous section. Since the arguments are similar to those in Section 2 we omit some of the details.

Proof of Theorem 1.12. Consider the sequences (a_i), (b_i), (c_i) and (d_i) as in Definition 2.21. Define

$$\boldsymbol{\Phi}(\sigma)=(\boldsymbol{u}\,|\,\boldsymbol{v}\,|\,\boldsymbol{w})=(u_1,u_2,\ldots,u_{n-r}\,|\,v_1,v_2,\ldots,v_r\,|\,w_1,w_2,\ldots,w_{m-r}), \tag{25}$$

where

- $u_i=$the number of b_j's smaller than a_i,
- $v_i=$the number of b_j's larger than b_i with $j<i$,
- $w_i=$the number of $(n-r+c_j)$'s larger than d_i.

When $r=m$, this mapping $\boldsymbol{\Phi}$ does not have the $\boldsymbol{w}$ component and is exactly the mapping introduced in Section 2. Mimic the terminology used in Section 2. We call $\boldsymbol{u}=\boldsymbol{u}(\sigma)$, $\boldsymbol{v}=\boldsymbol{v}(\sigma)$, and $\boldsymbol{w}=\boldsymbol{w}(\sigma)$, the *column inversion number sequence*, the *essential inversion number sequence* and the *row inversion number sequence*, respectively. We call $|\boldsymbol{u}(\sigma)|:=\sum u_i$, $|\boldsymbol{v}(\sigma)|:=\sum v_i$, and $|\boldsymbol{w}(\sigma)|:=\sum w_i$ the *column inversion number* of σ, the *essential inversion number* of σ and the *row inversion number* of σ respectively. It is easy to check using Definition 2.21 that $\boldsymbol{\Phi}(\sigma)$ satisfies the inequalities in Definition 1.3: Since the a_i's are strictly increasing, if $i_1<i_2$ then the number of b_j's smaller than a_{i_1} is less than or equal to the number of b_j's smaller than a_{i_2}. So $\boldsymbol{u}$ is increasing. Note that $\boldsymbol{b}$ is of length r. Thus

$$0\leqslant u_1\leqslant\cdots\leqslant u_{n-r}\leqslant r.$$

Similarly, we have

$$0\leqslant w_{m-r}\leqslant\cdots\leqslant w_1\leqslant r.$$

Since v_i is equal to the number of inversions in $\boldsymbol{b}$ of the form (b_i,b_j),

$$0\leqslant v_i\leqslant i-1,\quad 1\leqslant i\leqslant r.$$

Thus $\boldsymbol{\Phi}$ is a mapping from $R^r_{m,n}$ to $\Omega^r_{m,n}$. From (18) we have $l(\sigma)=l(P(\sigma))$. Since $\boldsymbol{a}$ and $\boldsymbol{d}$ are both increasing, every inversion of $P(\sigma)$ is of the form (a_i,b_j) or (b_i,b_j) or $(n-r+c_j,d_i)$. Thus,

$$l(P(\sigma))=\sum_{i=1}^{n-r}u_i+\sum_{j=1}^{r}v_j+\sum_{k=1}^{m-r}w_k.$$

This proves (8).

Now we need to show that $\boldsymbol{\Phi}$ is a bijection. By Lemma 2.23, $|\Omega^r_{m,n}|=|R^r_{m,n}|$. Thus we need only show that $\boldsymbol{\Phi}$ is surjective.

Here, we use Algorithm 2.24. Suppose we have a sequence $(\boldsymbol{u}|\boldsymbol{v}|\boldsymbol{w})$ from $\Omega^r_{m,n}$. First, use (1) of the algorithm on $\boldsymbol{u}$. We get a strictly increasing sequence $\boldsymbol{a}=(a_1,a_2,\dots,a_{n-r})$. Then, use (2) of the algorithm on $\boldsymbol{a}$ and $\boldsymbol{v}$. Thus we get the sequence $\boldsymbol{b}=(b_1,b_2,\dots,b_r)$. Next, in a similar way, use (1) of the algorithm on $n-w_1,n-w_2,\dots,n-w_{m-r}$. We get a strictly increasing sequence $\boldsymbol{d}=(d_1,d_2,\dots,d_{m-r})$, where $d_i=n-w_i+i$. Arrange the elements of $[n+m-r]-[n-r]-\{d_1,d_2,\dots,d_{m-r}\}$ to form a strictly increasing sequence $\boldsymbol{c}=(n-r+c_1,n-r+c_2,\dots,n-r+c_r)$. Define a permutation of $[m+n-r]$ by

$$P=\left(\begin{array}{c|c|c} 1\,2\,3\cdots n-r & \boldsymbol{c} & \boldsymbol{d} \\ \boldsymbol{a} & \boldsymbol{b} & n+1\;\; n+2\cdots n+m-r \end{array}\right).$$

Think of this permutation as a permutation matrix. If we delete the first $n-r$ rows and the last $m-r$ columns from P, we get some $\sigma\in R^r_{m,n}$ such that $P(\sigma)=P$. Hence $\boldsymbol{\Phi}(\sigma)=(\boldsymbol{u}|\boldsymbol{v}|\boldsymbol{w})$. Thus $\boldsymbol{\Phi}$ is surjective. □

We give an example to illustrate the argument used in the proof. Since we have already shown how to see Algorithm 2.24 in Example 2.25 we choose a particular triple $(\boldsymbol{u}|\boldsymbol{v}|\boldsymbol{w})$ and give the corresponding $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ and $\boldsymbol{d}$.

Example 3.26. Let $m=n=4$ and $r=2$. Then $m+n-r=6$. For $(\boldsymbol{u}|\boldsymbol{v}|\boldsymbol{w})=(1,2\,|\,0,1\,|\,1,0)$, we have $\boldsymbol{a}=(2,4)$, $\boldsymbol{b}=(3,1)$, $\boldsymbol{c}=(3,5)$ and $\boldsymbol{d}=(4,6)$. Hence

$$P(\sigma)=\left(\begin{array}{cc|cc|cc} 1 & 2 & 3 & 5 & 4 & 6 \\ 2 & 4 & 3 & 1 & 5 & 6 \end{array}\right)$$

$$=\left(\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 0 \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & 1 \end{array}\right).$$

The submatrix in the southwest corner is σ.

4. The poset $(R^r_{n^m},\leqslant)$

In this section, we define a partial order on $R^r_{n^m}$. Recall that we identify the symmetric group W_k with the group of $k\times k$ permutation matrices.

Definition 4.27. Define a graph with $R^r_{n^m}$ as its vertex set as follows. If $\sigma, \tau \in R^r_{n^m}$ and $\sigma \neq \tau$, we say that σ and τ are adjacent if there exists some $s \in S(m)$ such that $s\sigma = \tau$ or there exists some $s' \in S(n)$ such that $\sigma s' = \tau$ where $S(k) = \{(12), (23), \ldots, (k-1, k)\}$.

Since $S(k)$ generates W_k and $W_m \times W_n$ acts transitively on $R^r_{n^m}$, by means of left and right multiplications, the graph is connected.

Definition 4.28. Let l be defined as in Definition 1.4. For $\sigma, \tau \in R^r_{nm}$ define $\sigma \leqslant \tau$ if $\sigma = \tau$ or there exists a sequence of elements $\sigma_1, \sigma_2, \ldots, \sigma_k \in R^r_{n^m}$ such that $\sigma = \sigma_1$, $\tau = \sigma_k$, each σ_i is adjacent to σ_{i+1} in the graph and $l(\sigma_i) < l(\sigma_{i+1})$, for $i \in [k-1]$. If $\sigma = v_r = \sum_{i=1}^r E_{i,n-r+i}$, then we call the sequence $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ above a reduced sequence of τ.

Lemma 4.29. *$(R^r_{n^m}, \leqslant)$ is a partially ordered set with minimal element v_r.*

Proof. This is immediate from the definition. $\square$

Definition 4.30. The level set of rank k in $(R^r_{n^m}, \leqslant)$ is defined by

$$L_k(R^r_{n^m}) = \{\sigma \in R^r_{n^m} \mid l(\sigma) = k\}.$$

In Corollary 1.16, we stated without proof that for an m by n rectangular board,

$$RL_r(n^m, q) = \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} [r]!. \tag{26}$$

Now we give a proof of this corollary.

Proof of Corollary 1.16. Let $Y_{n,r}$ be the set of all (right justified) Ferrers boards contained in an $n-r$ by r matrix. The set of sequences $\boldsymbol{u} = (u_1, \ldots, u_{n-r})$ satisfying $0 \leqslant u_1 \leqslant \cdots \leqslant u_{n-r} \leqslant r$ may be put in one-to-one correspondence with $Y_{n,r}$: the Ferrers board corresponding to $\boldsymbol{u}$ has columns of length $u_1, \ldots, u_{n-r}$. For example, if $n = 7$ and $r = 3$, then the sequence

$$(u_1, u_2, u_3, u_4) = (0, 1, 1, 3)$$

corresponds to the Ferrers board

$$\begin{matrix} * & * & * \\ & & * \\ & & * \end{matrix}$$

which is contained in a 3×4 matrix.

Clearly, the inclusion relation of Ferrers boards is a ranked partial order on the set $Y_{n,r}$ where the rank of the partition corresponding to $\boldsymbol{u}$ is $|\boldsymbol{u}| = \sum_{i=1}^{n-r} u_i$. This ranked poset has the following generating function:

$$\sum_{u\in Y_{n,r}} q^{|u|}=\begin{bmatrix} n \\ r \end{bmatrix}. \tag{27}$$

This formula has many proofs. See for example [14, 5]. Similar is the case for the sequence $(\boldsymbol{w})$. By Hall's Theorem, we know that the set of r-tuples $\boldsymbol{v}=(v_1,\dots,v_r)$ satisfying $0\leqslant v_i\leqslant i-1$, for $1\leqslant i\leqslant r$ is in one-to-one correspondence with W_r and that $|\boldsymbol{v}|=\sum_{i=1}^r v_i$ is the number of inversions in the permutation corresponding to $\boldsymbol{v}$. Thus,

$$\sum_{w\in W_r} q^{l(w)}=\sum_{0\leqslant v_i\leqslant i-1} q^{|v|}.$$

Thus by Theorem 1.12,

$$\begin{aligned}\sum_{\sigma\in R^r_{n^m}} q^{l(\sigma)} &= \sum_{\sigma\in R^r_{n^m}} q^{|u(\sigma)|+|v(\sigma)|+|w(\sigma)|}\\ &=\left(\sum_{u\in Y_{n,r}} q^{|u|}\right)\left(\sum_{0\leqslant v_i\leqslant i-1} q^{|v|}\right)\left(\sum_{w\in Y_{n,r}} q^{|w|}\right)\\ &=\left(\sum_{u\in Y_{n,r}} q^{|u|}\right)\left(\sum_{\omega\in W_r} q^{l(\omega)}\right)\left(\sum_{w\in Y_{n,r}} q^{|w|}\right)\\ &=\begin{bmatrix} n \\ r \end{bmatrix}[r]!\begin{bmatrix} m \\ r \end{bmatrix}.\end{aligned}$$

This is the formula in our corollary. $\square$

Corollary 4.31. *The sequence $(|L_k(R^r_{n^m})|)_{k\geqslant 0}$ of cardinalities of level sets is symmetric and unimodal.*

Proof. Since the coefficients of $[{k \atop r}]$ and $[r]!$ are both symmetric and unimodal (see [1, 19, 15]), the coefficients of $\sum_{\sigma\in R^r_{n^m}} q^{l(\sigma)}$ are symmetric and unimodal by Corollary 1.16. $\square$

Lemma 4.32. *Let $\lambda=(\lambda_1,\dots,\lambda_m)$ be a partition with $\lambda_1=n$. Then, R^r_λ is an order ideal of $R^r_{n^m}$.*

Proof. Suppose, arguing by contradiction, that for a fixed $\sigma\in R^r_\lambda$ there exists some $\tau\leqslant\sigma$ such that $\tau\notin R^r_\lambda$. Choose such τ so that $l(\tau)$ is maximum. Then, by the definition of our poset $(R^r_{n^m},\leqslant)$, there exists a sequence of elements $\tau_0,\tau_1,\dots,\tau_t\in R^r_{n^m}$ such that $\tau=\tau_0,\sigma=\tau_t$, each τ_i is adjacent to τ_{i+1}, and $l(\tau_i)<l(\tau_{i+1})$ for $i\in[t-1]$. Since $l(\tau)$ is maximum, $\tau_i\in R^r_\lambda$, for all $i>0$. Since τ_i is adjacent to τ_{i+1}, there exists some $s\in S(m)$ such that $s\tau=\tau_1$ or some $s'\in S(n)$ such that $\tau s'=\tau_1$. Without loss of generality, we consider the first case, only. Then there exists some k such that $s=(k,k+1)$ and $s\tau=\tau_1$. The assumption that $\tau\notin R^r_\lambda$ and $\tau_1\in R^r_\lambda$ implies the following:

- there exists a rook in the $(k+1, n-h)$ position of τ such that $h > \lambda_{k+1}$, i.e, in τ, this rook is outside the board F_λ.
- if the kth row of τ is nonzero, the rook in the kth row is to the right of the $(n-h)$th column since otherwise, $\tau_1 \notin R^r_\lambda$.

Therefore, $l(\tau) > l(\tau_1)$ by the equality in Theorem 1.12 whether the kth row is zero or not. Thus, we get a contradiction. $\square$

5. Further evaluation of length function

Lemma 5.3. *Suppose λ and μ are partitions and $\sigma \in R^r_\lambda \cap R^r_\mu$. Then $l_\lambda(\sigma) = l_\mu(\sigma)$.*

Proof. Write $l_\lambda(\sigma)$ and $l_\mu(\sigma)$ for the two length functions defined on R^r_λ and R^r_μ according to Definition 1.4. We want to prove that they are equal for $\sigma \in R^r_\lambda \cap R^r_\mu$. We may assume that $\mu = (n^m)$. Then $l_\mu(\sigma) = l(\sigma)$ as in Sections 2–4. For any given $\sigma \in R^r_\lambda$, if

$$\sigma = s_k s_{k-1} \cdots s_2 s_1 v_r s'_1 s'_2 \cdots s'_{h-1} s'_h,$$

with $k+h$ being minimum, then Lemma 4.32 tells us that each

$$s_{k'} s_{k'-1} \cdots s_2 s_1 v_r s'_1 s'_2 \cdots s'_{h'-1} s'_{h'} \in R^r_\lambda,$$

for $1 \leqslant k' \leqslant k$, $1 \leqslant h' \leqslant h$. Thus, each reduced sequence of σ in R^r_λ is also a reduced sequence of σ in $R^r_{n^m}$. Hence, our equality $l_\lambda(\sigma) = l(\sigma)$ is true. $\square$

The length function l can be evaluated 'locally' by the following formula which counts the contribution of each rook in σ, individually.

Proposition 5.34 (Local Formula). *Let $\sigma \in R^r_\lambda$. Write $\sigma = \sum_{i=1}^r E_{c_i, b_i}$ where $c_1 < c_2 < \cdots < c_r$. Let α_i be the number of zero rows above the c_i-th row in σ, γ_i the number of zero columns to the right of the b_i-th column in σ, and β_i the number of 1's to the 'northeast' of the i-th 1 (not including the i-th 1). Then,*

$$l(\sigma) = \sum_{i=1}^{r} (\alpha_i + \beta_i + \gamma_i). \tag{28}$$

Proof. Here we use (8). Clearly, $v_i = \beta_i$, for each i. Counting the set of all the pairs of the form (rook, zero column) where the rook is to the left of the zero column, we have

$$\begin{aligned}\sum_{i=1}^{n-r} u_i &= \sum_{i=1}^{n-r} \text{number of rooks to the left of the } i\text{th zero column in } \sigma \\ &= \sum_{i=1}^{r} \text{number of zero columns to the right of the } i\text{th rook} \\ &= \sum_{i=1}^{r} \alpha_i.\end{aligned}$$

Similarly, we have $\sum_{i=1}^{m-r} w_i = \sum_{i=1}^{r} \gamma_i$. $\square$

As an immediate consequence of the Local Formula 5.34 we have the following recurrence formula.

Corollary 5.35. *Let* $\lambda=(\lambda_1,\ldots,\lambda_m)$ *be a partition. Let* λ_{m+1} *be an integer such that* $1\leqslant\lambda_{m+1}\leqslant\lambda_m$. *Then*

$$RL_r(\lambda\cup\lambda_{m+1,q})=RL_r(\lambda,q)+q^{m-r+1}(\lambda_{m+1})_q RL_{r-1}(\lambda-1^m,q),$$

where $\lambda-1^m:=(\lambda_1-1,\ldots,\lambda_m-1)$.

Remark. If some of the entries in $\lambda-1^m$ are zero we agree to omit them so that $\lambda-1^m$ is a partition.

Proof of Corollary 5.35. Consider a rook placement σ on $F_{\lambda\cup\lambda_{m+1}}$. If there is no rook in the $(m+1)$st row, this rook placement can be viewed as a rook placement on F_λ. These placement contribute the first term of $RL_r(\lambda\cup\lambda_{m+1},q)$. Consider placements σ which have a rook in the $(m+1)$st row. Separate these according to the column containing the rth rook. Suppose the rth rook lies in the $(k+1)$st columns from the right where $0\leqslant k\leqslant\lambda_{m+1}-1$ and k is fixed for the moment. We compute $\alpha_r+\beta_r+\gamma_r$ in the local formula 5.34. Since there are $m-r+1$ zero rows above this rook, $\alpha_r=m-r+1$. We cannot compute β_r and γ_r separately but we can compute $\beta_r+\gamma_r$. This is the number of rooks to the northeast plus the number of zero columns to the right of the rth rook. This is equal to the number k of columns to the right of the rth ro ok . Let σ' be the rook placement on $F_{\lambda-1^m}$ consisting of the $r-1$ rooks in the first m rows of σ. Then

$$l(\sigma)=l(\sigma')+\alpha_r+\beta_r+\gamma_r=l(\sigma')+(m-r+1)+k.$$

Thus for fixed k the contribution of these placements σ to $RL_r(\lambda\cup\lambda_{m+1},q)$ is $q^{m-r+1}\,q^k\,RL_{r-1}(\lambda-1^m,q)$. Now sum over k to get the second term on the right-hand side of our formula. $\square$

Remark. The recurrence formula in Corollary 5.35 is not equivalent to Garsia–Remmel's recurrence formula. In fact, it reduces the size of the Ferrers boards faster than the formula in Theorem 1.9 does. We use an example to make a comparison. Let $(\lambda_1,\lambda_2,\lambda_3)=(4,3,3)$ and $r=2$. By Corollary 5.35

$$\begin{aligned}RL_2((4,3,3),q)&=RL_2((4,3),q)+q^{2-2+1}(3)_q RL_1((3,2),q)\\&=0+q^0(3)_q RL_1((3),q)+q(3)_q(RL_1((3),q)+q^{(1-1+1)}(2)_q)\\&=(3)_q(3)_q+q(3)_q((3)_q+q(2)_q)\\&=1+3q+6q^2+7q^3+5q^4+2q^5.\end{aligned}$$

By Theorem 1.9,

$$\begin{aligned}
R_2((4,3,3),q) &= q^{3-2}R_2((3,2,2),q)+(3-2+1)_qR_1((3,2,2),q)\\
&= q(q^{3-2}R_2((2,1,1),q)+(3-2+1)_qR_1((2,1,1),q))\\
&\quad +(2)_q(q^{3-1}R_1((2,1,1),q)+(3-1+1)_qR_0((2,1,1),q))\\
&= q(q(0+(3-2+1)_qR_1((1),q))+(q(2)_q+q^2(2)_q)(q^{3-1}R_1((1),q)\\
&\quad +(3-1+1)_qR_0((1),q))+(3)_q(2)_qR_0((2,1,1),q)\\
&= q^2(2)_q+(q(2)_q+q^2(2)_q)(q^2+(3)_qq)+(3)_q(2)_qq^4\\
&= q^7+3q^6+6q^5+7q^4+5q^3+2q^2.
\end{aligned}$$

Note that $R_2((4,3,3),q)=q^{C_{\lambda,2}}RL_2((4,3,3),q^{-1})$, where

$$C_{\lambda,2}=4+3+3-2\cdot 3/2=7.$$

This checks (10) in the example.

Let $\{(c_i,b_i)\}_{1\le i\le r}$ be as in Corollary 5.34. In [18] it was proved, by using the root system for W_n, that on an n by n square board

$$l(\sigma)=\sum_{i=1}^{r}((c_i-1)+n-b_i)+\mathrm{Inv}(b_1,\ldots,b_r)-r(r-1). \tag{29}$$

We will prove that the formula (29) is true on arbitrary Ferrers boards.

Lemma 5.36. *Let $\sigma\in R_\lambda^r$. Then,*

$$l(\sigma)=\sum_{i=1}^{r}((c_i-1)+n-b_i)+\mathrm{Inv}(b_1,\ldots,b_r)-r(r-1).$$

Proof. Since the rook placements on a Ferrers board do not have a corresponding root system we use the local formula. We will show that the right-hand side of formula (29) equals the right-hand side of the local formula 5.34. Note that

$$\mathrm{Inv}(b_1,\ldots,b_r)=\sum_{i=1}^{r}\beta_i.$$

Thus we need only show that

$$r(r-1)+\sum_{i=1}^{r}(\alpha_i+\gamma_i)=\sum_{i=1}^{r}((c_i-1)+n-b_i). \tag{30}$$

Note that we can obtain the right-hand side of (30) as follows: Move the ith rook of σ to the north until it reaches the top row. Then, move the rook to the right until it reaches the northeast corner of the Ferrers board. The number of steps is $(c_i-1)+(n-b_i)$. In this process (1) the ith rook in σ passes every nonzero row above the c_ith row, and every nonzero column to the right of the b_ith column and (2) the ith rook in σ passes through all the zero rows above the c_ith row and all the zero columns to the right of the b_ith column. Let k_i be the number of nonzero rows above row c_i and h_i be the number of zero columns to the right of column b_i. Then $c_i-1=\alpha_i+k_i$ and $n-b_i=\gamma_i+h_i$. Hence,

$$\sum_{i=1}^{r}((c_i-1)+n-b_i)=\sum_{i=1}^{r}(\alpha_i+\gamma_i+k_i+h_i).$$

Since $(k_1,\ldots,k_r)$ and $(h_1,\ldots,h_r)$ are two permutations of the set $[r]$ we have

$$\sum_{i=1}^{r}h_i=\sum_{j=1}^{r}k_j=\binom{r}{2}.$$

Hence, the lemma is proved. □

Proof of Theorem 1.13. By Lemma 4.32, all rook placements of F_λ with r rooks can be obtained by adjacent row interchanges and adjacent column interchanges from the 'canonical' rook placement v_r in such a way that all the intermediate rook placements stay inside F_λ. So we need only show that $GR(\sigma)+l(\sigma)$ is an invariant under the action of the transposition $s_i=(i,i+1)$ on the rows and on the columns. We prove that this is true for the action of s_i on the rows. The situation concerning the columns is exactly the same. We separate four cases.

(a) the ith and the $(i+1)$th row are both nonzero,

(b) the ith row is nonzero and the $(i+1)$th row is zero,

(c) the ith row is zero and the $(i+1)$th row is nonzero,

(d) the ith and the $(i+1)$th row are both zero.

In case (a), there are two subcases.

(a1) The ith rook is to the right of the $(i+1)$th rook.

(a2) The ith rook is to the left of the $(i+1)$th rook.

Suppose we are in case (a1). We show that $GR(s_i\sigma)=GR(\sigma)+1$. By Definition 1.5, we need to show that the number of ∘'s in the configuration of $s_i\sigma$ is one more than the number of ∘'s in the configuration of σ. Let k and j be the column indices of the ith and the $(i+1)$th rook, respectively. Then, $j<k$. Consider the two row array formed by the ith row and the $(i+1)$th row.

$$\begin{array}{ccccccc} & (j) & & (k) & \\ \cdots & \bullet & \cdots & 1 & \cdots \\ \cdots & 1 & \cdots & \bullet & \cdots \end{array}$$

By Definition 1.5, if there is a column of the form $\substack{\bullet\\ \bullet}$ in this array, either all columns of the form $\substack{\bullet\\ \bullet}$ are to the right of the kth column, i.e., they are determined by the ith rook and the $(i+1)$th rook or they are both determined by a rook in some row below the $(i+1)$th row. In both of the cases, the action of s_i does not change this column $\substack{\bullet\\ \bullet}$. If there is a column of the form $\substack{\circ\\ \circ}$ in the array, the column must be to the left of the jth column. Again, this column will not be affected by the action of s_i. If there is a column of the form $\substack{\bullet\\ \circ}$ then the dot $\bullet$ on the top is determined by the rook of the ith row according to Definition 1.5, which implies that the column is to the right of the kth column. But, this means the bottom cannot be a $\circ$, by Definition 1.5. So, such a column does not exist. If there is a column of the form $\substack{\circ\\ \bullet}$ the column must be between the jth and the kth columns. Thus, the bottom dot is determined by the $(i+1)$th rook, only. Hence, the action of s_i changes this column into the form $\substack{\bullet\\ \circ}$. Finally, the jth column $\substack{\bullet\\ 1}$ is changed into the form $\substack{1\\ \circ}$ and the kth column $\substack{1\\ \bullet}$ is changed into the form $\substack{\bullet\\ 1}$ which increases the number of $\circ$'s by one. Thus the number of $\circ$'s in the configuration of $s_i\sigma$ is one more than that in the configuration of σ so $GR(s_i\sigma)=GR(\sigma)+1$. On the other hand, the local formula 5.34 tells us that $l(s_i\sigma)=l(\sigma)-1$. So the sum $GR(\sigma)+l(\sigma)$ is not affected by the action of s_i.

By a similar argument, we can prove that the sum of l and GR is not affected by the action of s_i in the cases (a2), (b) and (c). Clearly, in case (d), the action of s_i does not change $GR(\sigma)$ or $l(\sigma)$. So, the sum $GR(\sigma)+l(\sigma)$ is an invariant under permutations of the rows of σ. Thus, $GR(\sigma)+l(\sigma)$ is a constant $C_{\lambda,r}$ depending only on λ and r. Since $l(v_r)=0$, by Definition 1.5,

$$C_{\lambda,r}=GR(v_r)=\sum_{i=1}^{l}\lambda_i-\frac{r(r+1)}{2}.$$

Therefore, Theorem 1.13 is proved. $\square$

Remark. Thus we have five different ways to evaluate our length function l. They are given by Definition 1.4, Theorem 1.12, formula (29), local formula 5.34 and Theorem 1.13. The fourth is the only one which calculates the length function, 'locally', i.e., it counts the contribution of each rook in σ, individually. This local property is very useful in the proof of Theorem 1.14.

6. The formula of rook length polynomials

In this section we prove the formula of rook length polynomials in Theorem 1.14. The main tool in this proof is the local formula 5.34 for the length function l which enables us to look at the contribution of each rook to the generating function, individually. We begin with the case $r=m$ which is Corollary 1.17. This says that

$$RL_m(\lambda,q)=(\lambda_1-m+1)_q(\lambda_2-m+2)_q\cdots(\lambda_{m-1}-1)_q(\lambda_m)_q$$

$$=\prod_{j=1}^{m}(\lambda_j-m+j)_q. \tag{31}$$

Suppose $m=1$. Then $\lambda=(\lambda_1)$ has one part so there is one rook and $\alpha_1=\beta_1=0$. If the rook lies in the kth column from the right then $\gamma_1=k-1$. Then $l(\sigma)=\gamma_1=k-1$. Thus $RL_1(\lambda,q)=\sum_{k=1}^{\lambda_1}q^{k-1}=(\lambda_1)_q$. Now we use induction on m. Choose k with $1\leqslant k\leqslant\lambda_m$. Fix k temporarily and consider the set of placements in which the rook in the last row (i.e., the mth row) is in the kth column from the right. Since $r=m$, there are no zero rows. Thus $\alpha_i=0$ for $1\leqslant i\leqslant m$ and the rook in the mth row is the mth rook. Consider a fixed placement σ. Suppose in the first $m-1$ rows there are $t=t(\sigma)$ rooks to the northeast of the mth rook. Thus there are $k-1-t$ zero columns to the right of the mth rook. So $\beta_m=t$ and $\gamma_m=k-1-t$. Thus $\alpha_m+\beta_m+\gamma_m=k-1$ is independent of t. Let $\lambda'=(\lambda_1-1,\dots,\lambda_{m-1}-1)$ be the partition which has Ferrers board obtained from F_λ by deleting the row and column containing the mth rook. Let σ' be the placement on the board $F_{\lambda'}$ obtained from σ by deleting the m-th rook. Note that σ' is a placement of $m-1$ rooks on a Ferrers board with $m-1$ rows. By the Local Formula 5.34 $l(\sigma)=l(\sigma')+k-1$. Thus the contribution of these placements to $RL_m(\lambda,q)$ is

$$\sum_{\sigma'}q^{l(\sigma')+k-1}=q^{k-1}RL_{m-1}(\lambda',q).$$

By the induction hypothesis,

$$RL_{m-1}(\lambda',q)=\prod_{j=1}^{m-1}((\lambda_j-1)-(m-1)+j)_q.$$

Now varying k we get

$$RL_m(\lambda,q)=\sum_{k=1}^{\lambda_m}q^{k-1}\prod_{j-1}^{m-1}((\lambda_j-1)-(m-1)+j)_q,$$

which is formula (31).

Finally, we consider the general case $r\leqslant m$. Fix indices $1\leqslant i_i<\cdots<i_r\leqslant m$ and consider placements σ such that the r rooks live in the rows with indices $i_1,\dots,i_r$. Let $F_{\lambda'}$ be the board obtained from F_λ by deleting the rows different from $i_1,\dots,i_r$. Let σ' be the placement of r rooks obtained by restricting σ to the board $F_{\lambda'}$. We use the local formula to express $l(\sigma)$ in terms of $l(\sigma')$. Note that the β_j and γ_j for σ' are the same as for σ. Thus

$$l(\sigma)=l(\sigma')+\sum_{j=1}^{r}\alpha_j. \tag{32}$$

Now we compute the α_j. Since there are i_1-1 zero rows above the first rook, $\alpha_1=i_1-1$. Define $i_0=0$. In general, there are $\alpha_j:=\sum_{u=1}^{j}(i_u-i_{u-1}-1)$ zero rows above the jth rook for $1 \leqslant j \leqslant r$. Hence

$$\sum_{j=1}^{r} \alpha_j = \sum_{j=1}^{r} \sum_{u=1}^{j} (i_u - i_{u-1} - 1)$$
$$= \sum_{j=1}^{r} (r-j+1)(i_j - i_{j-1} - 1).$$

It follows from (32) that the contribution to $RL_r(\lambda, q)$ from the placements σ such that the r rooks live in the rows with indices $i_1, \ldots, i_r$ is

$$q^{\sum_{j=1}^{r}(r-j+1)(i_j - i_{j-1} - 1)} \prod_{j=1}^{r} (\lambda_{i_j} - r + j)_q.$$

Now sum over all the indices $i_1, \ldots, i_r$ to get the theorem. □

Proof of Corollary 1.19. Let λ be a parabolic board of type $(\mu_1, \ldots, \mu_k)$. Note that $m = \sum_{i=1}^{k} \mu_i$. If $1 \leqslant j \leqslant m$, write $j = \sum_{i=1}^{t-1} \mu_i + s$, where $1 \leqslant s \leqslant \mu_t$ and $1 \leqslant t \leqslant k$. Then $\lambda_j - m + j = s$ for $1 \leqslant s \leqslant \mu_t$ and $1 \leqslant t \leqslant k$. Therefore, by (31), we have

$$RL_m(\lambda, q) = (\lambda_1 - m + 1)_q (\lambda_2 - m + 2)_q \cdots (\lambda_{m-1} - 1)_q (\lambda_m)_q = \prod_{i=1}^{k} [\mu_i]!. \qquad \square$$

Comments. The results obtained here have other consequences. For example, Corollary 1.16 implies unimodality of the coefficient sequence of the Garsia–Remmel polynomial $R_r(\lambda, q)$ and the rook length polynomial $RL_r(\lambda, q)$ for a rectangular board. On the other hand, Corollary 1.19 implies unimodality of the coefficient sequence of the Garsia–Remmel polynomial $R_m(\lambda, q)$ and rook length polynomial $RL_m(\lambda, q)$ on any Ferrers board with m rows. These strongly support the conjecture made by Garsia and Remmel that the sequence of coefficients of $R_r(\lambda, q)$ is unimodal for arbitrary λ and r.

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